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Quantum superintegrable systems for arbitrary spin

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Abstract

In Pronko and Stroganov (1977 *Zh. Eksp. Teor. Fiz.* **72** 2048, 1997 *Sov. Phys.—JETP* **45** 1072) the superintegrable system which describes the magnetic dipole with spin $\frac{1}{2}$ (neutron) in the field of linear current was considered. Here we present its generalization for any spin which preserves superintegrability. The dynamical symmetry stays the same as it is for spin $\frac{1}{2}$.

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1. Introduction

There exist few quantum systems where the degeneration of spectrum is bigger when it follows from geometrical symmetry of the problem. The famous examples of such systems are isotropic oscillator, Kepler problem, rotator and some other which have no physical interpretation. This supplementary degeneration of the spectrum arises due to dynamical symmetry (which includes trivial geometrical). In this way, the geometrical symmetry $SO(3)$ extends to the group $SU(3)$ in the case of isotropic oscillator and to the group $SO(4)$ in the case of bound spectrum of Kepler problem. These kinds of systems are also called maximally superintegrable. Their characteristic property is that all finite trajectories are closed. Thirty years ago, with Stroganov, we had found another example of the physical system which possesses supplementary degeneration of its spectrum due to the existence of hidden symmetry. The system describes the magnetic dipole with spin $\frac{1}{2}$ (neutron) in the field of line current. The obvious, geometrical symmetry is the symmetry $SO(2)$ with respect to rotation around the z-axis, the direction of current (the translation along z is trivially separated). Dynamical group in this case is $SO(3)$. Here we are speaking about the symmetry for the negative part of the spectrum. For scattering states, this group changes as in the case of the Kepler problem and becomes the other real form of complex $SO(3)$, namely $SO(2, 1)$ (or $E(2)$ for $E = 0$).

The peculiarity of the system which we discovered is that it describes the particle with spin, that was not known before. The question which was raised soon after is whether it is possible to preserve dynamical symmetry for the particles with higher spins. The answer up to now was negative in spite of many attempts [2–4]. The previous considerations failed, because people wanted to preserve the interaction of the spin particle with the external field which corresponded to intuitive picture. But the truth is that the particle with higher spin may interact not only by its dipole magnetic moment, but also through higher moments as well. For example, the particle with spin 1 acquires the possibility to have apart from dipole also quadruple interaction, for spin $\frac{3}{2}$, octuple interaction etc[1]. Certainly, this modification of interaction does not describe any longer an elementary particle like neutron in the magnetic field of the linear wire. At the same time, the emerged system could be useful for the description of trapped ultra-cold atoms in this field—the subject was intensely discussed in the literature [5, 6]. Apparently atoms, being extended objects will manifest its structure in inhomogeneous magnetic field through interaction which is much more complicated with simple dipole interaction of neutron. As we shall see below, the requirement of maximal superintegrability fixes the form of interaction up to a finite number of parameters—for spin s the number of parameters is $2s + 1$. Accidentally or not, but almost all the known maximally superintegrable systems have a wide physical application. The system which we considered in [1] was discussed in [7] in connection with the trapping of ultra-cold neutrons. Hence it is possible that the interaction we found for higher spins also includes some important cases.

The Hamiltonian of the system, considered in [1], is given by

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2m} - \mu \mathbf{H}, \tag{1}$$

where μ is the magnetic moment of the particle and $\mathbf{H}$ is the magnetic field of linear current directed along the z-axis:

$$\mathbf{H} = CI\left(\frac{y}{r^2}, -\frac{x}{r^2}\right). \tag{2}$$

The constant coefficient C depends on the unit system, in practical system $C = 0.2$. Thus, the final form of the Hamiltonian will be

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2m} - k\frac{s_x y - s_y x}{r^2}, \tag{3}$$

where the coefficient k collected all constants. For spin $\frac{1}{2}$ the spin operator is proportional to Pauli matrices. The Hamiltonian (1) is invariant with respect to rotations around the z-axis generated by $J_z = L_z + s_z$. In addition to this geometrical integral, the Hamiltonian (3) possesses two non-trivial

$$\begin{aligned} A_x &= \frac{1}{2}(J_3 p_x + p_x J_3) + km\frac{s_x y - s_y x}{r^2} y, \\ A_y &= \frac{1}{2}(J_3 p_y + p_y J_3) - km\frac{s_x y - s_y x}{r^2} x. \end{aligned} \tag{4}$$

The integrals (4) together with the Hamiltonian and J_z form the following algebra:

$$\begin{aligned} &[J_z, A_x] = \mathrm{i}A_y, \qquad [J_z, A_y] = -\mathrm{i}A_x, \qquad [A_x, A_y] = -\mathrm{i}\mathcal{H}J_z, \\ &[A_x, \mathcal{H}] = 0, \qquad [A_y, \mathcal{H}] = 0. \end{aligned} \tag{5}$$

If we now define the operators

$$J_x = A_x(-\mathcal{H})^{-1/2}, \qquad J_y = A_y(-\mathcal{H})^{-1/2}, \tag{6}$$

then the following commutation relations of $SO3$ algebra hold true:

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk} J_k. \tag{7}$$

[1] As a matter of fact, the importance of other interaction for higher spins manifests itself also in case of the Heisenberg magnetic, which is integrable only if the interaction between spins is modified.

While we designed the operators J_i, we had in mind the discrete spectrum for which the energy is negative. For positive energy, the algebra will be $SO(2,1)$, because some signs in (7) will change. The Casimir operator of algebra (7) is expressed via the Hamiltonian

$$\mathbf{J}^2 = J_1^2 + J_2^2 + J_3^2 = -\frac{1}{4} - \frac{mk^2}{2\mathcal{H}}; \tag{8}$$

therefore, the Hamiltonian is given by

$$\mathcal{H} = -\frac{mk^2}{2}\frac{1}{\mathbf{J}^2 + \frac{1}{4}}. \tag{9}$$

The representations of $SO(3)$ are characterized by the integer or half-integer spin. In our problem, it is clear that the eigenvalues of J_3 could be only half-integer due to addition of spin $\frac{1}{2}$ and integer orbital momentum, therefore only half-integer representations will be realized. So the eigenvalues of $\mathbf{J}^2$ will be $\frac{2n+1}{2}\left(\frac{2n+1}{2}+1\right)$, $n = 0, 1, \ldots$, and the spectrum of energy will be

$$E_n = -\frac{mk^2}{2}\frac{1}{(n+1)^2}. \tag{10}$$

The supplementary degeneration in this case means that the spectrum does not depend on the eigenvalue of J_z.

The existence of additional integrals of motion in this case is based completely on the algebraic properties of Pauli matrices which represent spin $\frac{1}{2}$ operators for it and direct substitution of it; matrices which represent any other spin immediately destroy the whole construction. The exit of this situation we will discuss in the following section.

2. High spins

Let us consider the quantum system which describes the neutral particle with arbitrary spin s in the field of rectilinear electric current. The general form of the Hamiltonian in this case is given by the following equation (here we omitted the trivial part of kinetic motion along the line of the current):

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m} + \frac{M(\mathbf{s},\mathbf{x})}{\mathbf{x}^2}, \tag{11}$$

where $\mathbf{p}$, $\mathbf{x}$ are the two-dimensional vectors and $\mathbf{s}$ is the spin operator, $\mathbf{s} = (s_x, s_y, s_z)$. The matrix $M(\mathbf{s},\mathbf{x})$ will be specified later. Now we shall impose on $M(\mathbf{s},\mathbf{x})$ only one condition:

$$[M(\mathbf{s},\mathbf{x}), J_z] = 0, \tag{12}$$

where $J_z = L_z + s_z$. Now let us look for the additional integrals of motion in the following form:

$$A_i = \tfrac{1}{2}(p_i J_z + J_z p_i) + V_i(\mathbf{s},\mathbf{x}), \tag{13}$$

where $V_i(\mathbf{s},\mathbf{x})$ is an operator which should be defined by interaction $M(\mathbf{s},\mathbf{x})$. The commutator of the first term of (13) with Hamiltonian (11) gives

$$\left[\mathcal{H}, \frac{1}{2}(p_i J_z + J_z p_i)\right] = \frac{\mathrm{i}}{2}\left\{J_z, \partial_i \frac{M(\mathbf{s},\mathbf{x})}{\mathbf{x}^2}\right\}, \tag{14}$$

where $\{A, B\} = AB + BA$. This form of commutator suggests the following structure of the operator $V_i(\mathbf{s},\mathbf{x})$:

$$V_i(\mathbf{s},\mathbf{x}) = \epsilon_{ij}\frac{x_i}{\mathbf{x}^2}M(\mathbf{s},\mathbf{x}), \tag{15}$$

where ϵ_{ij} is the antisymmetric tensor. Indeed, commuting (15) with the Hamiltonian we obtain

$$\left[\mathcal{H}, \epsilon_{ij}\frac{x_i}{\mathbf{x}^2}M(\mathbf{s},\mathbf{x})\right] = -\frac{\mathrm{i}}{2m}\left[-\left\{L_z, \partial_i \frac{M(\mathbf{s},\mathbf{x})}{\mathbf{x}^2}\right\} + \left\{p_k, \epsilon_{ik}\left(\frac{M(\mathbf{s},\mathbf{x})}{\mathbf{x}^2} + x_j\partial_j\frac{M(\mathbf{s},\mathbf{x})}{\mathbf{x}^2}\right)\right\}\right], \tag{16}$$

Now if we add and subtract s_z to L_z, we can rewrite (16) in the following form:

$$\left[\mathcal{H}, \epsilon_{ij}\frac{x_i}{\mathbf{x}^2}M(\mathbf{s},\mathbf{x})\right] = -\frac{\mathrm{i}}{2m}\left[-\left\{J_z, \partial_i \frac{M(\mathbf{s},\mathbf{x})}{\mathbf{x}^2}\right\} + \partial_i\left\{s_z, \frac{M(\mathbf{s},\mathbf{x})}{\mathbf{x}^2}\right\} + \left\{p_k, \epsilon_{ik}\left(\frac{M(\mathbf{s},\mathbf{x})}{\mathbf{x}^2} + x_j\partial_j\frac{M(\mathbf{s},\mathbf{x})}{\mathbf{x}^2}\right)\right\}\right]. \tag{17}$$

Imposing on the matrix $M(\mathbf{s},\mathbf{x})$ apart from (12), the conditions

$$s_z M(\mathbf{s},\mathbf{x}) + M(\mathbf{s},\mathbf{x})s_z = 0, \qquad \left(\frac{M(\mathbf{s},\mathbf{x})}{\mathbf{x}^2} + x_j\partial_j\frac{M(\mathbf{s},\mathbf{x})}{\mathbf{x}^2}\right) = 0, \tag{18}$$

we arrive at the commutativity of

$$A_i = \frac{1}{2}(p_i J_z + J_z p_i) - m\epsilon_{ij}\frac{x_i}{\mathbf{x}^2}M(\mathbf{s},\mathbf{x}) \tag{19}$$

with the Hamiltonian. Note that the matrix $M(\mathbf{s},\mathbf{x})$, which we had in the previous section for spin $\frac{1}{2}$, satisfies both conditions (18) and this was the reason why we achieved the commutativity of integrals (4) with the Hamiltonian. Now it is possible to prove that the commutation relations for the components of A_i are

$$[A_i, A_j] = -\mathrm{i}\epsilon_{ij}J_z 2m\mathcal{H}, \tag{20}$$

provided the same conditions (12) and (18) are satisfied.

Now let us take care of the matrix $M(\mathbf{s},\mathbf{x})$. The second equation (18) is rather simple and it requires $M(\mathbf{s},\mathbf{x})$ to be a homogenous function of x_i of degree 1. So we can present $M(\mathbf{s},\mathbf{x})$ in the form

$$M(\mathbf{s},\mathbf{x}) = |\mathbf{x}|\mu(\mathbf{s},\mathbf{n}), \qquad \mathbf{n} = \frac{\mathbf{x}}{|\mathbf{x}|}, \tag{21}$$

where the matrix $\mu(\mathbf{s},\mathbf{n})$ commutes with J_z and anticommutes with s_z. Let us consider these conditions in the basis $|s,k\rangle$ of the unitary representation of spin s. This basis is defined by

$$\begin{aligned} &s_z|s,k\rangle = k|s,k\rangle, \qquad \mathbf{s}^2|s,k\rangle = s(s+1)|s,k\rangle,\\ &s_+|s,k\rangle = \sqrt{s(s+1)-k(k+1)}|s,k+1\rangle,\\ &s_-|s,k\rangle = \sqrt{s(s+1)-k(k-1)}|s,k-1\rangle,\\ &k = s, s-1,\ldots,-s. \end{aligned} \tag{22}$$

In this basis, $\mu(\mathbf{s},\mathbf{n})$ has its matrix elements $\mu_{kk'}(\mathbf{n})$,

$$\mu_{kk'}(\mathbf{n}) = \langle s,k|\mu(\mathbf{s},\mathbf{n})|s,k'\rangle. \tag{23}$$

The first equation (18) implies the following:

$$(k+k')\mu_{kk'}(\mathbf{n}) = 0. \tag{24}$$

The solution of this equation is

$$\mu_{kk'}(\mathbf{n}) = \delta_{k,-k'}a_k(\mathbf{n}), \qquad a_k^*(\mathbf{n}) = a_{-k}(\mathbf{n}), \tag{25}$$

where the last condition guarantees that $\mu(\mathbf{s}, \mathbf{n})$ will be Hermitian. Now let us impose the condition (12) on matrix $\mu(\mathbf{s}, \mathbf{n})$:

$$[J_z, \mu(\mathbf{s}, \mathbf{n})] = 0 \Rightarrow [L_z, a_k(\mathbf{n})] + 2ka_k(\mathbf{n}) = 0. \tag{26}$$

This equation fixes the $\mathbf{n}$-dependence of $a_k(\mathbf{n})$:

$$a_k(\mathbf{n}) = \alpha_k\, \mathrm{e}^{-2\mathrm{i}k\varphi}, \qquad \mathrm{e}^{\mathrm{i}\varphi} = n_1 + \mathrm{i}n_2, \qquad \alpha_k^* = \alpha_{-k}. \tag{27}$$

So, the final expression for the matrix $\mu_{kk'}(\mathbf{n})$,

$$\mu_{kk'}(\mathbf{n}) = \delta_{k,-k'}\alpha_k\, \mathrm{e}^{-2\mathrm{i}k\varphi}, \tag{28}$$

contains $2s + 1$ real parameters which define the set of α_k. The matrix $\mu_{kk'}(\mathbf{n})$ could also be expressed in terms of operators $\mathbf{s}$:

$$\begin{aligned}\mu(\mathbf{s}, \mathbf{n}) = {} & (\beta_s(s_+n_-)^{2s} + \mathrm{h.c.}) + (\beta_{s-1}(s_z - s)(s_+n_-)^{2s-2}(s_z + s) + \mathrm{h.c.}) \\ & + (\beta_{s-2}(s_z - s)(s_z - s + 1)(s_+n_-)^{2s-4}(s_z + s)(s_z + s - 1) + \mathrm{h.c.}) \cdots .\end{aligned} \tag{29}$$

The structure of this expression could be understood from the following explanations. First, note that due to condition (26), the matrix $\mu(\mathbf{s}, \mathbf{n})$ can depend only on the combinations of (s_+n_-), (s_-n_+) and s_z. Second, the matrix $\mu(\mathbf{s}, \mathbf{n})$ in representation (22) is anti-diagonal and in order to obtain an operator which has nonzero matrix elements in the upper and lower corners, we need to take a linear combination of (s_+n_-) and its conjugated in maximal power—for spin s it is $2s$. In this way, we obtain the first term of (29). The second term is obtained with the same strategy but here we need to eliminate the action of $(s_+n_-)^{2s-2}$ on the vectors $|-s, s\rangle$ and $\langle s, s|$. This explains the appearance of the fringe multipliers $(s_z - s)$ and $(s_z + s)$. The rest is just a repetition of this procedure. The parameters β_k in (29) play the same role, as α_k in (27) but only $\beta_s = \alpha_s$, the others are different because of additional multipliers, depending on s_z in (29).

It is interesting that even for $s = \frac{1}{2}$, we have not only one type of interaction, which respects dynamical symmetry, but two. Indeed, according to the present consideration, the Hamiltonian

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m} + a\frac{s_x y - s_y x}{r^2} + b\frac{s_x x + s_y y}{r^2} \tag{30}$$

also possesses dynamical symmetry. The additional term in this Hamiltonian describes electric dipole in electric field $\frac{\vec{r}}{r^2}$ which is produced by rectilinear charge.

The last issue which we are going to discuss is the analogue of formula (9) in the generic case. Defining as in (6) the operators J_i, having in mind discrete spectrum, we obtain

$$\mathbf{J}^2 + \frac{1}{4} = -\frac{m}{2}\frac{\mu(\mathbf{s}, \mathbf{n})^2}{\mathcal{H}}. \tag{31}$$

The matrix $\mu(\mathbf{s}, \mathbf{n})$ commutes with the Hamiltonian, as it follows from its definition and the form of $\mathcal{H}$ (11), and from (31) we obtain

$$\mathcal{H} = -\frac{m}{2}\frac{\mu(\mathbf{s}, \mathbf{n})^2}{\mathbf{J}^2 + \frac{1}{4}}. \tag{32}$$

As the matrix $\mu(\mathbf{s}, \mathbf{n})$ in the bases $|s, k\rangle$ is anti-diagonal, its square is diagonal

$$(\mu(\mathbf{s}, \mathbf{n})^2)_{kk'} = \mathrm{diag}\{|\alpha_s|^2, |\alpha_{s-1}|^2, \ldots, |\alpha_s|^2\}, \tag{33}$$

so it could be written as a linear combination of projectors on the states with definite values of s_z.

In conclusion, we summarize the above discussion. It is shown that the problem introduced in [1] for spin $\frac{1}{2}$ admits generalization for arbitrary spin which preserves dynamical symmetry.

The interaction depends on $2s+1$ parameters which leaves a big freedom for applications. The energy spectrum has the same character $1/n^2$ as in the case of spin $\frac{1}{2}$, but for the wavefunctions we need to select the proper representations of $SO(3)$ corresponding to the given value of spin. In [1], we have constructed explicitly the invariant form of the Schrödinger equation for our system as was done by Fock [8] for the Kepler problem. This form exists also for arbitrary spin. In the present paper, we have not touched the subject of possible applications of the system we considered, as this is the theme for a separate paper.

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